On the dimension of diagonally affine self-affine sets and overlaps.

Károly Simon

Department of Stochastics
Institute of Mathematics
Budapest University of Technology and Economics
www.math.bme.hu/~simonk

Joint work with Balázs Bárány and Michal Rams

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Co-authors

Bárány Balázs Univ. of Warwick/ TU Budapest (left)
Michal Rams, Warsaw IMPAN (right)

Abstract

In this talk we consider diagonally affine, planar IFS

$$\Phi = \{S_i(x,y) = (\alpha_i x + t_{i,1}, \beta_i y + t_{i,2})\}_{i=1}^m .$$

Combining the techniques of Hochman and Feng-Hu we compute the Hausdorff dimension of the self-affine attractor and measures and we give an upper bound for the dimension of the exceptional set of parameters.

Definitions

$$(1) \qquad \Phi = \{S_i(x,y) = (\alpha_i x + t_{i,1}, \beta_i y + t_{i,2})\}_{i=1}^{m},$$

where $0 < |\alpha_i|, |\beta_i| < 1$, and we assume that

$$S_i([0,1]^2) \subset [0,1]^2.$$

We call a Borel probability measure μ self-affine if it is compactly supported with support Λ and there exists a $\mathbf{p} = (p_1, \dots, p_m)$ probability vector such that

$$(2) \qquad \mu = \sum_{i=1}^{m} p_i \mu \circ S_i^{-1}.$$

Lyapunov exponent

The entropy and the Lyapunov exponents of μ:

$$\boxed{h_\mu} := -\sum_{i=1}^{m} p_i \log p_i,$$

and

$$\text{(3)} \qquad \boxed{\chi_\alpha} := -\sum_{i=1}^{m} p_i \log |\alpha_i|, \boxed{\chi_\beta} := -\sum_{i=1}^{m} p_i \log |\beta_i|.$$

Lyapunov exponents: $0 < \chi_1 \le \chi_2 \le \chi_3 \le \chi_4$

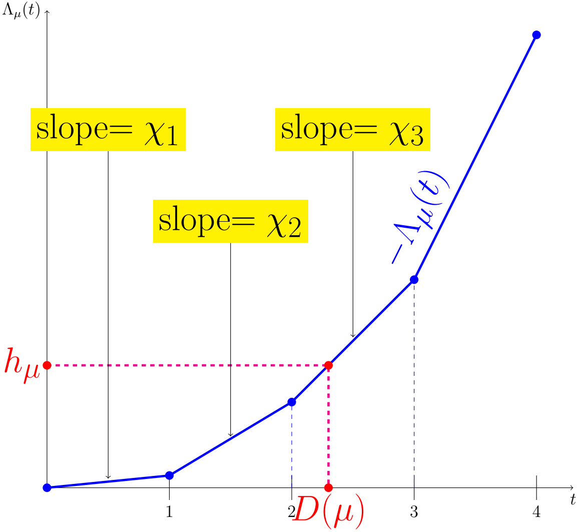


Figure: Definition of the Laypunov dimension $D(\mu)$

Lyapunov dimension by formula

If

(4)
$$k := \max\left\{i : 0 < h_\nu - \chi_1(\nu) - \cdots - \chi_i(\nu)\right\} \leq d - 1,$$

then we define the Lyapunov dimension of ν:

$$D(\nu) := k + \frac{h_\nu - \chi_1(\nu) - \cdots - \chi_k(\nu)}{\chi_{k+1}(\nu)};$$

If $h_\nu - \chi_1(\nu) + \cdots - \chi_d(\nu) > 0$ then we define

$$D(\nu) := d \cdot \frac{h_\nu}{\chi_1(\nu) + \cdots + \chi_d(\nu)},$$

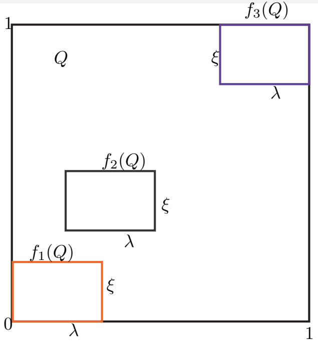
where h_ν is the entropy of the measure ν.

Lyapunov dimension on the plane

$$D(\nu) := \begin{cases} \dfrac{h_\nu}{\chi_1(\nu)}, & \text{if } h_\nu \le \chi_1(\nu); \\ 1 + \dfrac{h_\nu - \chi_1(\nu)}{\chi_2(\nu)}, & \text{if } \chi_1(\nu) < h_\nu \le \chi_1 + \chi_2(\nu); \\ 2 \cdot \dfrac{h_\nu}{\chi_1(\nu) + \chi_2(\nu)}, & \text{if } h_\nu > \chi_1(\nu) + \chi_2(\nu). \end{cases}$$

What does this mean exactly? See it in a special case:

Lyapunov dimension in a very special case



Let ν be the self-affine measure which corresponds to $p_1 = p_2 = p_3 = \frac{1}{3}$.
$\chi_1 = -\log\lambda < -\log\xi = \chi_2$
Clearly, $h_\nu = \log 3$

$$D(\nu) = \begin{cases} \dfrac{\log 3}{-\log\lambda}, & \text{if } \lambda < \frac{1}{3}; \\ \dfrac{\log 3 - \log\lambda}{-\log\xi}, & \text{if } \lambda > \frac{1}{3}; \end{cases}$$

An old result

First we consider an old result due to Balázs Bárány.

Notation

- $s_\alpha = \dim_{\mathrm{B}} \mathrm{proj}_x \Lambda$, $s_\beta := \dim_{\mathrm{B}} \mathrm{proj}_y \Lambda$.
- d_α and d_β are the solutions of the equations:

$$\sum_{i=1}^{m} \alpha_i^{s_\alpha} \beta_i^{d_\alpha - s_\alpha} = 1 \text{ and } \sum_{i=1}^{m} \beta_i^{s_\beta} \alpha_i^{d_\beta - s_\beta} = 1$$

Bárány's Theorem

Theorem 1.1 (Bárány (2011))

W.L.G. we may assume that $S_i([0,1]^2) \subset [0,1]^2$. *Assume that* $\{S_i\}_{i=1}^m$ *satisfies Strong Separation Condition:*

$$S_i([0,1]^2) \cap S_j([0,1]^2) = \emptyset. \tag{5}$$

Then

$$\dim_{\mathrm{B}} \Lambda = \max\{d_\alpha, d_\beta\}. \tag{6}$$

Hochman-condition

We say that an IFS

$$\mathcal{G} = \{f_i(x)\}_{i \in \mathcal{S}}$$

of similarities on the real line satisfies the Hochman-condition if there exists an $\varepsilon > 0$ such that for every $n > 0$

$$\min\left\{\Delta(\overline{\imath}, \overline{\jmath}) : \overline{\imath}, \overline{\jmath} \in \mathcal{S}^n,\ \overline{\imath} \neq \overline{\jmath}\right\} > \varepsilon^n,$$

where

$$\Delta(\overline{\imath}, \overline{\jmath}) = \begin{cases} \infty & f'_{\overline{\imath}}(0) \neq f'_{\overline{\jmath}}(0) \\ |f_{\overline{\imath}}(0) - f_{\overline{\jmath}}(0)| & f'_{\overline{\imath}}(0) = f'_{\overline{\jmath}}(0). \end{cases}$$

Examples when Hochman Condition holds

If the parameters of the IFS

$$\mathcal{G} = \{f_i(x) = r_i x + t_i\}_{i \in \mathcal{S}}$$

of similarities are algebraic, i.e.

both t_i and r_i are algebraic numbers ,

then either

- the Hochman-condition holds or
- there is a complete overlap, that is, there exist $n \geq 1$, and $\overline{\imath} \neq \overline{\jmath} \in \mathcal{S}^n$ such that

$$f_{\overline{\imath}}(0) = f_{\overline{\jmath}}(0)$$

.

Hochman Theorem

Suppose that an IFS $\Psi = \{r_i x + t_i\}_{i=1}^{m}$, $|r_i| < 1$ of contracting similarities on the real line satisfies the Hochman-condition. Let $\mathbb{P} := \{p_1, \dots, p_m\}^{\mathbb{N}}$. Then for the measure

$$\mu = \mathbb{P} \circ \Pi^{-1},$$

$$\dim_H \mu = \min\left\{1, \frac{h_\mu}{\chi}\right\},$$

where h_μ is the entropy and χ is the Lyapunov exponent:

$$h_\mu = -\sum_{i=1}^{M} p_i \log p_i \text{ and } \chi = -\sum p_i \log r_1$$

.

Hausdorff dimension of a measure

Here we recall the Hausdorff dimension of a probability measure μ,

$$\dim_H \mu = \inf\{\dim_H A : \mu(A) = 1\} = \operatorname*{ess\,sup}_{\mu \sim x} \liminf_{r \to 0+} \frac{\log \mu(B_r(x))}{\log r},$$

Families of self-similar IFS

Let $I \subset \mathbb{R}$ be a compact parameter interval and $m \geq 2$. For every parameter $t \in I$ given a *self − similar* IFS on the line:

$$\Phi_t := \{\varphi_{i,t}(x) = r_i(t) \cdot (x - a_i(t))\}_{i=1}^m ,$$

where

$$r_i : I \to (-1, 1) \setminus \{0\} \text{ and } a_i : I \to \mathbb{R}$$

are real analytic functions. Let Π_t be the natural projection from $\Sigma := \{1, \dots, m\}^{\mathbb{N}}$ to the attractor Λ_t of Φ_t.

Families of self-similar IFS (cont.)

For every probability vector $\mathbf{p} := (p_1, \ldots, p_m)$ the associated self-similar measure is

$$\nu_{\mathbf{p},t} := (\Pi_t)_*(\mathbf{p}^{\mathbb{N}}).$$

Its similarity dimension is defined by

$$\dim_{\mathrm{S}}(\nu_{\mathbf{p},t}) := \frac{\sum_{i=1}^{m} p_i \log p_i}{\sum_{i=1}^{m} p_i \log r_i(t)}$$

The similarity dimension of Λ_t is the solution $s(t)$ of

$$r_1^{s(t)}(t) + \cdots + r_m^{s(t)}(t) = 1.$$

Families of self-similar IFS (cont.)

We say that a parameter $t \in I$ is exceptional if either

$$\dim_{\mathrm{H}} \Lambda_t < \min\{1, s(t)\}$$

or there exists a probability vector $\mathbf{p} := (p_1, \dots, p_m)$ such that

$$\dim_{\mathrm{H}}(\nu_{\mathbf{p},t}) < \min\{1, \dim_{\mathrm{S}}(\nu_{\mathbf{p},t})\}.$$

Families of self-similar IFS (cont.)

Theorem 1.2 (Hochman)

Assume that for $\mathbf{i}, \mathbf{j} \in \Sigma = \{1, \ldots, m\}^{\mathbb{N}}$ *we have*

if $\Pi_t(\mathbf{i}) = \Pi_t(\mathbf{j})$ *holds for* ***all*** $t \in I$ *then* $\mathbf{i} = \mathbf{j}$.

Then both the Hausdorff and the packing dimension of the set of exceptional parameters are equal to 0.

Theorem A

Theorem A Let Φ be an IFS of the form (1) and let μ be a self-affine measure of the form (2). Without loss of generality we may assume that

$$\boxed{\chi_{\alpha} \leq \chi_{\beta}}$$

(i.e. the direction of *y*-axis is strong stable direction).

Theorem A (cont.)

1. Suppose Φ_α satisfies the Hochman-condition and $\frac{h_\mu}{\chi_\alpha} \leq 1$. Then

$$\dim_H \mu = \frac{h_\mu}{\chi_\alpha}.$$

2. Suppose Φ_α and Φ_β satisfy the Hochman-condition and $\frac{h_\mu}{\chi_\beta} \leq 1 < \frac{h_\mu}{\chi_\alpha}$. Then

$$\dim_H \mu = 1 + \frac{h_\mu - \chi_\alpha}{\chi_\beta}.$$

Road towards Theorem B

As a consequence of Theorem A we can calculate the dimension of the attractor.

Denote by s_α and s_β the similarity dimensions of the IFSs Φ_α and Φ_β respectively, i.e. s_α and s_β are the unique solutions of the equations

$$\sum_{i=1}^{m} |\alpha_i|^{s_\alpha} = 1, \text{ and } \sum_{i=1}^{m} |\beta_i|^{s_\beta} = 1. \tag{7}$$

Theorem B

Theorem B Let Φ be an IFS of the form (1) and let Λ be the attractor of Φ. Without loss of generality we may assume that $s_\beta \leq s_\alpha$.

Theorem B (cont.)

1. Suppose Φ_α satisfies the Hochman-condition and $s_\alpha \leq 1$. Then
$$\dim_H \Lambda = \dim_B \Lambda = s_\alpha.$$
2. Suppose Φ_α and Φ_β satisfy the Hochman-condition and $s_\beta \leq 1 < s_\alpha$. Then
$$\dim_H \Lambda = \dim_B \Lambda = d,$$
where d is the unique solution of
$$\sum_{i=1}^m |\alpha_i||\beta_i|^{d-1} = 1.$$

Proposition C

Proposition C

Let Φ be an IFS of the form (1). Let us assume that

$$\max_{i\neq j}\left\{|\alpha_i|+|\alpha_j|\right\}<1$$

and

$$\sum_{i=1}^{m}|\beta_i|\leq 1.$$

Then there exists a set $\mathcal{T}\subset\mathbb{R}^{2m}$ such that $\dim_P\mathcal{T}\leq 2m-2$ and for every $(t_{1,1},\dots,t_{m,1},t_{1,2},\dots,t_{m,2})\in\mathbb{R}^{2m}\setminus\mathcal{T}$ the statements of Theorem A and Theorem B hold.

We obtained these estimates by using the method of Fraser and Shmerkin.

Peres and Shmerkin showed that for every self-similar set in $\mathbb{R}$ or $\mathbb{R}^2$ for any $\varepsilon > 0$ there exists a self-similar set contained in the original one with dimension ε-close to the dimension of the original set such that the IFS satisfies strong separation condition (SSC) and the functions share a common contraction ratio. That is, the IFS is homogeneous.

We show that under the above conditions there exists a homogeneous self-affine set satisfying the strong separation condition which approximates the dimension of the original set from below.

Theorem D

For an IFS $\mathcal{G} = \{\psi_i\}_{i=1}^{M}$ we define the kth iterate by

$$\mathcal{G}^k = \{\psi_{i_1} \circ \cdots \circ \psi_{i_k}\}_{i_1,\dots,i_k=1}^{M}.$$

Theorem D Let Φ be an IFS of the form (1) and let Λ be the attractor of Φ. Without loss of generality we may assume that $s_\beta \leq s_\alpha$. Suppose that either

1. Φ_α satisfies the Hochman-condition and $s_\alpha \leq 1$,

or

2. Φ_α, Φ_β satisfy the Hochman-condition and $s_\beta \leq 1 < s_\alpha$.

Theorem D (cont.)

Then for every $\varepsilon > 0$ there exists a homogeneous affine IFS Ψ of the form

$$\Psi = \{ T_j(x,y) = (\alpha x + u_{j,1}, \beta y + u_{j,2}) \}_{j=1}^{k} \tag{8}$$

with attractor $\Gamma \subseteq \Lambda$ such that Ψ is a subsystem of some iterate of Φ and satisfies the SSC, i.e.
$T_i(\Gamma) \cap T_j(\Gamma) = \emptyset$ and

$$\dim_H \Lambda - \varepsilon = \dim_P \Lambda - \varepsilon = \dim_B \Lambda - \varepsilon$$
$$\leq \dim_H \Gamma = \dim_P \Gamma = \dim_B \Gamma.$$

Notation

First we recall here some results and notations of Feng and Hu. Let

$$\Psi = \{\psi_i\}_{i=1}^{M}$$

be a strictly contracting IFS mapping $[0,1]^d$ into itself. Let

$$\Sigma = \{1,\ldots,M\}^{\mathbb{N}}$$

be the corresponding symbolic space, σ the usual left-shift operator on Σ and let m be a σ-invariant ergodic measure on Σ.

Notation (cont.)

Let Π be the natural projection, i.e.

$$\Pi(i_0, i_1, \dots) = \lim_{n\to\infty} \psi_{i_0} \circ \cdots \circ \psi_{i_n}(\underline{0}).$$

Let

$$\mathcal{P} = \{[1], \dots, [M]\}$$

be the partition of Σ, where

$$[i] = \{\mathbf{i} \in \Sigma : i_0 = i\}$$

and denote by $\mathcal{B}$ the Borel σ-algebra of $\mathbb{R}^d$.

The projection entropy

We define the *projection entropy* *of m under* Π *with respect to* Ψ as

$$h_{\Pi}(m) := H_m(\mathcal{P} \mid \sigma^{-1}\Pi^{-1}\mathcal{B}) - H_m(\mathcal{P} \mid \Pi^{-1}\mathcal{B}),$$

where $H_m(\xi \mid \eta)$ denotes the usual conditional entropy of ξ given η.

Feng-Hu Theorem for self-similar IFS

Let Ψ be an IFS of similarities on the real line. Then

$$\dim_H \mu = \frac{h_\Pi(\mathbb{P})}{\chi},$$

where $\mu = \mathbb{P} \circ \Pi^{-1}$ and

$$\chi = -\sum_{i=1}^{M} p_i \log |\psi_i'(0)|$$

is the Lyapunov exponent.

Notation

Let us assume that the maps of the IFS $\Psi = \left\{\psi_i : [0,1]^d \mapsto [0,1]^d\right\}_{i=1}^{M}$ have the form

$$\psi_i(x_1, \dots, x_d) = \left(\rho_{1,i}x_1 + t_{1,i}, \dots, \rho_{d,i}x_d + t_{d,i}\right).$$

For a $\mathbb{P} = \{p_1, \dots, p_M\}^{\mathbb{N}}$ Bernoulli measure, denote the Lyapunov exponents by

$$\boxed{\chi_j} = -\sum_{i=1}^{M} p_i \log\left|\rho_{j,i}\right|.$$

Without loss of generality we may assume that

$$0 < \chi_1 \le \chi_2 \le \cdots \le \chi_d.$$

Notation (cont.)

Let Ψ_k be the IFS with functions restricted to the first k coordinates, i.e. $\Psi_k = \left\{\psi_i^k : [0,1]^k \mapsto [0,1]^k\right\}_{i=1}^M$, where

$$\psi_i^k(x_1, \dots, x_k) = \left\{\left(\rho_{1,i}x_1 + t_{1,i}, \dots, \rho_{k,i}x_k + t_{k,i}\right)\right\}_{i=1}^M.$$

Denote the natural projection w.r.t Ψ_k by Π_k. Moreover, let

$$\mu_k = \mathbb{P} \circ \Pi_k^{-1},$$

where $P = (p_1, \dots p_k)^{\mathbb{N}}$.

Feng-Hu Theorem

For every $1 \leq k \leq d$,

$$\dim_H \mu_k = \frac{h_{\Pi_1}(\mathbb{P})}{\chi_1} + \sum_{j=2}^{k} \frac{h_{\Pi_j}(\mathbb{P}) - h_{\Pi_{j-1}}(\mathbb{P})}{\chi_j}.$$

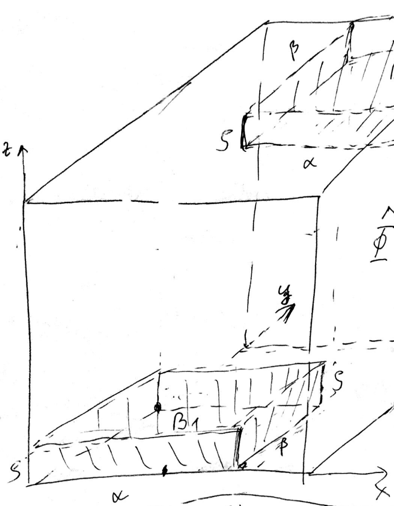
In particular, on the plane assuming SSC

(9)
$$\dim_{\mathrm{H}}(\mu) = \frac{h_\mu}{\chi_2(\mu)} + \left(1 - \frac{\chi_1(\mu)}{\chi_2(\mu)}\right) \cdot \dim_{\mathrm{H}}(\mu_x),$$

where $\mu_x = \mathrm{proj}_x \mu$. That is if SSC holds then
(10)

$$D(\mu) = \dim_{\mathrm{H}}(\mu) \iff \dim_{\mathrm{H}}(\mu_x) = \min\left\{1, \frac{h_\mu}{\chi_1(\mu)}\right\}.$$

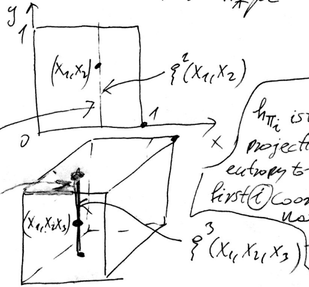
①



$\alpha > \beta > \gamma$

$\chi_1 = \log\frac{1}{\alpha} < \chi_2 = \log\frac{1}{\beta} < \chi_3 = \log\frac{1}{\gamma}$

$\mu = \{\frac{1}{2}, \frac{1}{2}\}^{\mathbb{N}}$ on Σ $\quad \nu := \Pi_* \mu$



h_{π_i} is the projection entropy to the first (i) coordinate.

$i = 1,2 \quad f_i([0,1]^3) = B_i$

Let $(\nu_2)_x^{\xi_2} :=$ conditional measure of ν_2 on $\xi^2(x_1, x_2)$

$(\nu_3)_x^{\xi_3} :=$ — || — $\nu_3 = \nu$ on $\xi^3(x_1, x_2, x_3)$

ν_1 is the proj of ν to x

ν_2 — || — (x, y) plane

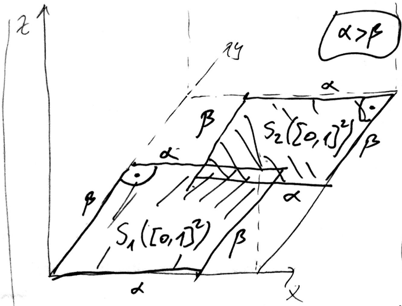
$\nu_3 = \nu$

$$\dim_H (\nu_2)_x^{\xi_2} = \frac{h_{\pi_2}(\mathbb{P}) - h_{\pi_1}(\mathbb{P})}{\chi_2}$$

$$\dim_H (\nu_3)_x^{\xi_3} = \frac{h_{\pi_3}(\mathbb{P}) - h_{\pi_1}(\mathbb{P})}{\chi_3}$$

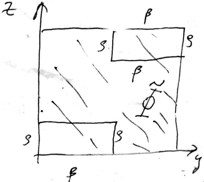
2

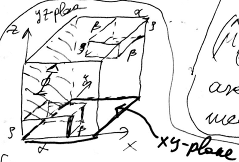
$\alpha > \beta$

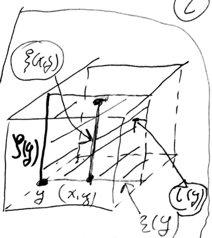


$S_i(x,y) = (\alpha_i x + t_{i,1}, \beta_i y + t_{i,2})$

$\Phi = \{S_i\}$







$\hat\mu^{\xi}_{(x,y)}, \hat\mu^{\tau}_y, \tilde\mu^{\varsigma}_y, \mu^{\xi}_y$ are the conditional measures.

$\hat\Phi = \{\hat S_i(x,y,z) = (\alpha_i x + t_{i,1}, \beta_i y + t_{i,2}; \varsigma z + t_{i,3})\}$

$\hat\Phi, \tilde\Phi$ satisfy SSC so from:

↳ Feng-Hu: $\dim_H \hat\mu = \frac{h_{\pi_\alpha}}{\chi_\alpha} + \frac{h_\pi - h_{\pi_\alpha}}{\chi_\beta} + \frac{h_{\hat\pi} - h_\pi}{-\log \varsigma}$ $\quad \dim_H \tilde\mu = \frac{h_{\pi_\beta}}{\chi_\beta} + \frac{h_\pi - h_{\pi_\beta}}{-\log\varsigma}$

Goal: $h_\pi = h_\mu$

$\dim \hat\mu^{\xi}_{(x,y)}$ for μ-a.a. (x,y)

$\min\{1, \frac{h_\mu}{\chi_\alpha}\} \overset{\text{Hochman cond}}{=} \dim_H \mu_\alpha \overset{\text{Feng-Hu}}{=} \frac{h_{\pi_\alpha}}{\chi_\alpha}$ | By assumption: $\frac{h_\mu}{\chi_\alpha} > 1 \Rightarrow \frac{h_{\pi_\alpha}}{\chi_\alpha} = 1 \Rightarrow \boxed{h_{\pi_\alpha} = \chi_\alpha}$

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